MATH 20D: MIDTERM 1 PRACTICE PROBLEMS

LECTURER: FINN MCGLADE UC SAN DIEGO

Below *NSS* is used to reference the text *Fundamentals of Differential Equations (9th edition) by Nagle, Saff, Snider*

Question (1). *Give a continuous explicit solution to each of the initial value problems below. In each case, be sure to give the domain of your solution.*

(a) $\frac{dy}{dx} = (1+y^2)\tan(x), \quad y(0) = \sqrt{3}$

(b) $\frac{1}{2}\frac{dy}{dx} = \sqrt{y+1}\cos(x), \quad y(\pi) = 0$

(c) $\frac{1}{\theta}\frac{dy}{d\theta} = \frac{y\sin\theta}{y^2+1}, \quad y(1) = 1$

(d) $\frac{dy}{dx} - \frac{y}{x} = xe^x, \quad y(1) = e^{-1}$

(e) $t^2\frac{dx}{dt} + 3tx = t^4\log(t) + 1, \quad x(1) = 0$

(f) $\cos(x)\frac{dy}{dx} + y\sin(x) = 2x\cos^2(x), \quad y\left(\frac{\pi}{4}\right) = \frac{-15\sqrt{2}\pi^2}{32}$

(g) $\frac{dy}{dx} = y^2 - 5y + 6, \quad y(0) = 1$

(h) $\frac{dy}{dx} = \log(y)\cos^6(x), \qquad y(0) = 1$

(i) $\frac{dy}{dt} = 2t\cos^2 y, \quad y(0) = \pi/4$

(j) $\sin(x)\frac{dy}{dx} + y\cos(x) = x\sin(x), \quad y\left(\frac{\pi}{2}\right) = 2$

(k) $\frac{dy}{dx} = y^2 - 5y + 6, \quad y(0) = 3$

(l) $\frac{dy}{dx} + 2y = \begin{cases} 0, & x < 0, \\ 1, & x \geq 0, \end{cases} \quad y(-1) = 0$

(m) $\frac{dy}{dx} = 8x^3e^{-2y}, \quad y(1) = 0$

(n) $\frac{dy}{dx} + \frac{2\cos(x)}{\sin(x)}y = \frac{e^x}{\sin(x)}, \quad y(\pi/2) = 1$

(o) $\frac{dT}{dt} = \begin{cases} 2(1-T), & t \leq \log(2), \\ e^t - \frac{1}{2}T, & t \geq \log(2), \end{cases} \quad T(0) = 0$

(p) $\frac{dN}{dt} = \begin{cases} \frac{1}{2}(1-N), & t \leq \log(4) \\ \frac{1}{2} - \frac{2N}{2-(t-\log(4))}, & t \geq \log(4) \end{cases}, \quad N(0) = 1$

Question (2). *Represent the solutions to the initial value problems below using a definite integral*

(a) $\frac{dy}{dx} = e^{x^2}y^2, \quad y(2) = 1$

(b) $\frac{dy}{dx} = \sqrt{1+\sin(x)}(1+y^2), \quad y(0) = 1$

(c) $\frac{dy}{dx} + \frac{\sin(2x)}{2(1+\sin^2(x))}y = 1, \quad y(0) = 0$

(d) $\frac{dy}{dx} + \sqrt{1+\sin^2(x)}y = x, \quad y(0) = 2$

(e) $\frac{dy}{dt} = \frac{1}{1+t^2} - y, \quad y(2) = 3$

(d) $\frac{dy}{dx} + y = \sqrt{1+\cos^2(x)}, \quad y(1) = 4$

Question (3). ***NSS 2.2.33*** *Suppose a brine containing 0.3 kilograms of salt per liter runs into a tank initially filled with 400 liters of water containing 2kg of salt. If the bring enters at 10L/min, the mixture is kept uniform by stirring, and the mixture flows out at the same rate. Find the mass of salt in the tank after 10 minutes.*

Question (4). ***NSS 3.2.4*** *A brine solution of salt flows at a constant rate of 4 L/min into a large tank that initially held 100 L of pure water. The solution inside the tank is kept well stirred and flows out of the tank at a rate of 3L/min. If the concentration of salt in the brine entering the tank*

Date:

is 0.2kg/L, determine the mass of salt in the tank after t min. When will the concentration of salt in the tank reach 0.1 kg/L.

Question (5)**.** *The air in a small 12 ft by 8ft by 8 ft room is 3% carbon monoxide. Starting at $t = 0$, fresh air containing no carbon monoxide is blown into the room at a rate of 100 ft^3/min.*

(a) If air in the room flows out through a vent at the same rate, calculate the volume of carbon monoxide in the room after 10 minutes.

At time $t = 10$ a gas leak develops and an additional 25ft^3 per minute of air at a concentration of 5% carbon monoxide begins to flow into the room.

(b) Calculate the volume of carbon monoxide in the room 20 minutes after the leak develops.

Question (6)**.** ***NSS 3.3.2*** *A cold beer initially at $35°F$ warms up to $40°F$ in 3 min while sitting in a room of termperature $70°F$. Assuming Newton's law of cooling predict how warm the beer will be if left out for 20 min? Give your answer to the nearest $0.01°F$.*

Question (7)**.** ***NSS 3.3.36*** *A pot of boiling water at $100°C$ is removed from a stove and left to cool in a kitchen. After 5 minutes, the water temperature has decreased to $80°C$, and after another 5 minutes it has dropped to $65°C$. Assuming the temperature of the kitchen is constant, use Newton's law of cooling to predict the temperature of the kitchen.*

Question. *At 4:00pm a customer at an indoor cafe is served a $95°C$ cup of coffee. Immediately after the coffee is served, the air conditioning unit in the cafe looses power. As a result, the temperature of the cafe changes in such a way that t minutes after 4:00pm, the temperature of the cafe is given by*

$$M(t) = 45 + (26 - 45)e^{-0.3t}.$$

Let $T(t)$ denote the temperature of the coffee t minutes after 4:00pm and assume

$$T'(t) = \frac{1}{5}(M(t) - T(t))$$

Calculate the temperature of the coffee at 4:25pm.

Question. *Determine whether the pairs of functions given below are linearly independent or linearly dependent on the domain $\mathbb{R}$.*

(a) $y_1(x) = \sin(x)$, $y_2(x) = \cos(x)$. ***(b)*** $y_1(x) = e^x$, $y_2(x) = e^{-x}$

(c) $y_1(x) = xe^x$, $y_2(x) = e^x$ ***(d)*** $y_1(x) = \arctan(x)$, $y_2(x) = \frac{1}{1+x^2}$

(e) $y_1(x) = \sin^2(x) + \cos^2(x)$, $y_2(x) = \sqrt{\pi}$ ***(f)*** $y_1(x) = 2x + 1$, $y_2(x) = x^3 + x$

(g) $y_1(x) = \cosh(2x)$, $y_2(x) = -4\sinh^2(x) - 2$ ***(h)*** $y_1(x) = x^5$, $y_2(x) = x^4$

Question (3)**.** *Write down general solutions to each of the ODE's below:*

(a) $2y'' + 7y' - 4y = 0$ ***(b)*** $y'' + y = 0$ ***(c)*** $y'' + 5y' + 6y = 0$

(d) $y'' - 4y' + 7y = 0$ ***(e)*** $6y'' + y' - 2y = 0$ ***(f)*** $y'' + 4y' + 8y = 0$

(g) $3y'' + 11y' - 7y = 0$ ***(h)*** $2y'' + 13y' - 7y = 0$ ***(i)*** $y'' + 6y' + 9y = 0$

(j) $z'' + 10z' + 25z = 0$ ***(k)*** $4y'' - 4y' + y = 0$ ***(l)*** $y'' - 2y' + 26y = 0$

Question (4)**.** *Suppose $a \neq 0$, b, and c are constant. Assume $y_1, y_2 \colon \mathbb{R} \to \mathbb{R}$ are linearly independent solutions to the ODE*

$$ay'' + by' + cy = 0 \tag{0.1}$$

and let $y(x) = C_1 y_1(x) + C_2 y_2(x)$ denote the corresponding general solution.
(a) Show that C_1 and C_2 satisfy the system of linear equations

$$\begin{cases} C_1 y_1(0) + C_2 y_2(0) = y(0) \\ C_1 y_1'(0) + C_2 y_2'(0) = y'(0) \end{cases}$$

(b) Suppose $b^2 - 4ac < 0$ and consider the linearly independent solutions to (0.1) given by

$$y_1(x) = e^{\alpha x}\cos(\beta x) \qquad \text{and} \qquad y_2(x) = e^{\alpha x}\sin(\beta x)$$

where $\alpha = -b/2a$ and $\beta = \sqrt{4ac - b^2}/2a$. Calculate $y_1(0)$, $y_2(0)$, $y_1'(0)$, and $y_2'(0)$. Then apply the result of (a) to deduce that

$$C_1 = y(0) \qquad \text{and} \qquad C_2 = -\frac{1}{\beta}(\alpha y(0) - y'(0)). \tag{0.2}$$

(c) Solve the initial value problems below:

(i) $y'' + 2y' + 2y = 0; \quad y(0) = 2, \quad y'(0) = 1$ ***(ii)*** $y'' + 2y' + 17y = 0; \quad y(0) = 1, \quad y'(0) = -1$

(iii) $2y'' - 4y' + 4y = 0; \quad y(0) = 0, \quad y'(0) = 1$ ***(iv)*** $y'' + 9y = 0; \quad y(0) = 1, \quad y'(0) = 1$

(v) $y'' - 2y' + 2y = 0; \quad y(\pi) = e^{\pi}, \quad y'(\pi) = 0$ ***(vi)*** $2y'' - 2y' + y = 0; \quad y(0) = 1, \quad y'(0) = -2$

(d) Rewrite each of your solutions to part (c) in the form

$$y(t) = Ae^{\alpha t}\sin(\beta t + \phi)$$

where $A > 0$ and $\phi \in [0, 2\pi)$.

Question. ***NSS 4.9.2*** *A 3-kg mass is attached to a spring with stiffness coefficient $k = 48$ N/m. The mass is displaced 1/2 m to the left of the equilibrium point and given a velocity of 2 m/sec to the right. Assuming the force of friction is negligible. find the equation of motion of the mass along with the amplitude, period, and frequency. How long after release does the mass pass through the equilibrium position?*

Question. ***NSS Chapter 4 Review Problem 38*** *A 3-kg mass is attached to a spring with stiffness coefficient $k = 75$ N/m. The mass is displaced 1/4 m to the left of the equilibrium point and given a velocity of 1 m/sec to the right. Assuming the force of friction is negligible. find the equation of motion of the mass along with the amplitude, period, and frequency. How long after release does the mass pass through the equilibrium position?*

Question. ***NSS 4.9.7*** *A 1/8 kg-mass is attached to a spring with stiffness 16 N/m. The damping constant for the system is 2N-sec/m. If the mass is moved 3/4 m to the left of equilibrium and given an initial velocity of 2m/sec determine the equation of motion of the mass. Express your solution in the form $y(t) = Ae^{\alpha t}\sin(\beta t + \phi)$.*

Question. ***NSS 4.9.10*** *A 1/4-kg mass is attached to a spring with stiffness 8 N/m. The coefficient of friction for the system is $b = 1/4$ N-sec/m. If the mass is moved 1 m to the left of equilibrium and released, what is the maximum displacement to the right that it will attain.*

Question. ***NSS 4.9.11*** *A 1 kg mass is attached to a spring with stiffness 100 N/m. The coefficient of friction for the system is 0.2 N-sec/m. If the mass is pushed rightward from the equilibrium position with a velocity of 1 m/sec, when will it attain its maximum dispacement to the right?*